Substructures Of [M,N]

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Received 25/11/2012
Accepted 04/03/2013

ABSTRACT

Let M_R, N_R be two modules over a ring R and $[M,N]=hom_R(M,N)$, then $[M,N]$ is an $(E_N,E_M)-$bimodule. The concern is about the five substructures of $hom_R(M,N)$: the Jacobson radical $J[M,N]$, the singular ideal $\Delta[M,N]$, the co-singular ideal $\nabla[M,N]$, the total $\mathrm{Tot}[M,N]$ and the $I[M,N]$. One natural question is to characterize when the total is equal to one or more of the other structures. Toward this question, many results have been obtained:

(a) ***A module W_R is locally projective if and only if, $\mathrm{Tot}[M,W]=\nabla[M,W]$ for every quasi-projective module $M\in \mathrm{mod}-R$.***

(b) ***A module Q_R is locally injective if and only if, $\mathrm{Tot}[Q,N]=\Delta[Q,N]$ for every quasi-injective module $N\in \mathrm{mod}-R$.***

(c) ***A module N_R with $\Gamma(N)=\{0\}$ is an $I-$module if and only if, $\mathrm{Tot}[M,N]=I[M,N]$ for every module $M\in \mathrm{mod}-R$.***

(d) ***A module N_R with $\Gamma(N)=\{0\}$ is an $I-$module if and only if, for every $a\in E_N$ with $Im(1-a)\subseteq J(N)$, is one-to-one.***

Keywords: *$hom_R(M,N)$, Semipotent ring, Jacobson radical, (co)singular ideal, the total.*

Mathematical Subject, Classification (MSC 2010): Primary: 16D10, 16N20, 16S50; Secondary: 16D40, 16D50.

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البنـــى الجزئيـــة لــ $[M, N]$

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تاريخ الإيداع 2012/11/25
قبل للنشر في 2013/03/04

الملخص

ليكن M_R و N_R مودولين فوق الحلقة $\boldsymbol{R}$ و $[M,N]=hom_R(M,N)$ عندئــذ $[M,N]$ يكون $(E_N,E_M)-$ مودولاً ثنائياً. هدفنا هو دراسة البنى الجزئية الخمــس فــي $hom_R(M,N)$ وهي: جذر جاكبسون $J[M,N]$، والمثالي المفرد $\Delta[M,N]$، والمثــالي المــزدوج $\nabla[M,N]$، والتوتال $\mathrm{Tot}\,[M,N]$، والمثالي $I[M,N]$. أحد الأسئلة الطبيعية هي توصيف الحالة التــي مــن أجلها يكون التوتال مساوياً إحدى البنى الجزئية الخمس السابقة. في معرض الإجابة عن هذا التساؤل تــم الحصول على عدد من النتائج الجديدة أهمها:

1. الـــشرط الـــلازم والكـــافي كـــي يكـــون المـــودول W_R مـــودولاً إســـقاطياً محليـــاً هـــو أن يكون $\mathrm{Tot}\,[M,W]=\nabla[M,W]$ وذلك لأجل أي مودول شبه إسقاطي $M\in \mathrm{mod}-R$.
2. الـــشرط الـــلازم والكـــافي كـــي يكـــون المـــودول Q_R مـــودولاً أفقيـــاً محليـــاً هـــو أن يكـــون $\mathrm{Tot}\,[Q,N]=\Delta[Q,N]$ وذلك لأجل أي مودول شبه أفقي $N\in \mathrm{mod}-R$.
3. ليكن N_R مودولاً من أجله $\Gamma(N)=\{0\}$. الشرط اللازم والكافي كي يكون المودول N_R هـــو $I-$مـــودول هـــو أن يكـــون $\mathrm{Tot}\,[M,N]=I[M,N]$ ؛ وذلـــك لأجـــل أي مـــودول $M\in \mathrm{mod}-R$.
4. ليكن N_R مودولاً من أجله $\Gamma(N)=\{0\}$. الشرط اللازم والكافي كي يكون المودول N_R هـــو $I-$مودول هو أن يتحقق الشرط: أياً كان $a\in E_N$ الذي من أجلـــه $Im(1-a)\subseteq J(N)$، فإن التشاكل a هو تشاكل واحد لواحد.

الكلمات المفتاحية: $hom_R(M,N)$، الحلقات شبه الجامدة، جذر جاكبسون، المثالي المنفرد، المثالي المزدوج، التوتال.

التصنيف الرياضي العالمي للعام (2010): أولاً: *16D10, 16N20, 16S50;*. ثانياً: *16D40* و *16D50*.

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1. Introduction.

In this paper rings R are associative with identity unless otherwise indicated. All modules over a ring R are unitary right modules. A submodule N of a module M is said to be small in M if $N + K \neq M$ *for any proper submodule* K *of* M *, [5]. A submodule N of a module M is said to be large (essential) in M if* $N \cap K \neq 0$ *for any nonzero submodule K of M, [5]. If* M *is an* $R-$*module, the radical of M denoted by* $J(M)$ *is defined to be the intersection of all maximal submodules of M. Also,* $J(M)$ *coincides with the sum of all small submodules of M. It happen that* M *has no maximal submodules in which case* $J(M) = M$ *,[9]. Thus, for a ring* R*,* $J(R)$ *is the Jacobson radical of R. For a submodule N of a module M, we use* $N \subseteq^{\oplus} M$ *to mean that* N *is a direct summand of M, and we write* $N \leq_e M$ *and* $N << M$ *to indicate that* N *is a large, respectively small, submodule of M. If* M_R *is a module, we use the notation* $E_M = End_R(M)$ *and we write* $I(E_M) = \{a : a \in E_M;\ Im(a) \subseteq J(M)\}$*. It is well known that* $I(E_M)$ *is an ideal in* E_M*, [5].*

2. Substructures of Hom.

Following [3,12,13], let M_R, N_R *are modules and* $[M,N] = hom_R(M,N)$*, then* $[M,N]$ *is an* $(E_N, E_M)-$*bimodule. The bimodule* $[M,N]$ *has three radicals: the Jacobson radical of* $[M,N]_{E_M}$*, the Jacobson radical of* ${}_{E_N}[M,N]$ *and the important Jacobson radical denoted by* $J[M,N]$ *defined by*

- *The Jacobson radical*

$$J[M,N] = \{a : a \in [M,N];\ b\,a \in J(E_M) \text{ for all } b \in [N,M]\}$$

$$J[M,N] = \{a : a \in [M,N];\ a\,b \in J(E_N) \text{ for all } b \in [N,M]\}$$

Thus, $J[M,M] = J(E_M)$*. In particular,* $J[R,R] = J(R)$*.*

- *The total*

$\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ a\ [N,M]$ *contains no nonzero idempotents*$\}$
$\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ [N,M]a$ *contains no nonzero idempotents*$\}$

Thus,

$\mathrm{Tot}[M,M]=\mathrm{Tot}(E_M)=\{a : a\in E_M;\ a\ E_M$ *contains no nonzero idempotents*$\}$
$=\{a : a\in E_M;\ E_M\ a$ *contains no nonzero idempotents*$\}$.

- *The singular ideal* $\Delta[M,N]=\{a : a\in[M,N];\ Ker(a)\leq_e M\}$.

In particular, $\Delta[M,M]=\Delta E_M$.

- *The co-singular ideal* $\nabla[M,N]=\{a : a\in[M,N];\ Im(a)<<N\}$.

In particular, $\nabla[M,M]=\nabla E_M$.

- *Following[3],we use the nation* $I[M,N]=\{a : a\in[M,N];Im(a)\subseteq J(N)\}$.

In particular, $I(E_M)=I[M,M]=\{a : a\in E_M;Im(a)\subseteq J(M)\}$.

It is easy to see that $J[M,N]$, $\nabla[M,N]$, *and* $\Delta[M,N]$*are contained in* $\mathrm{Tot}[M,N]$.

In particular, $J(E_M)$, $\nabla(E_M)$ *and* $\Delta(E_M)$ *are contained in* $\mathrm{Tot}(E_M)$.

In the study of the total, one of the interesting questions is when the total equals the Jacobson radical. Toward this question, many results have been obtained for module M which satisfies: every its submodule not contained in J(M) contains a nonzero projective direct summand of M. We start with the following lemma:

Lemma 2.1. *Let* M_R, N_R *be modules. Then:*

(1) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ b\,a\in\mathrm{Tot}(E_M)$ *for all* $b\in[N,M]\}$.

(2) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ a\,b\in\mathrm{Tot}(E_M)$ *for all* $b\in[N,M]\}$.

Proof. (1). *Let* $a\in\mathrm{Tot}[M,N]$. *suppose that* $b\,a\notin\mathrm{Tot}(E_M)$ *for some* $b\in[N,M]$ *then there exists* $g\in E_M$ *such that* $0\neq g(ba)=[g(ba)]^2\in E_M$. *Since* $g\,b\in[N,M]$ *then*

$0 \neq g\,(b\,a) = [(gb)\,a]^2 \in [N,M]a$, *a contradiction. Let* $a \in [M,N]$ *such that* $b\,a \in \mathrm{Tot}\,(E_M)$ *for all* $b \in [N,M]\}$. *Suppose that* $a \notin \mathrm{Tot}\,[\mathrm{M},\mathrm{N}]$ *then* $[N,M]a$ *contains a nonzero idempotent. So there exists* $g \in [N,M]$ *such that* $0 \neq g\,a = (g\,a)^2 \in E_M$ *and* $g\,a \in (g\,a)E_M$, *so* $g\,a \notin \mathrm{Tot}\,(E_M)$, *a contradiction. Similarly* (2) *holds.*

In particular, if for a ring R, Tot $(R) = J(R)$, *then a ring* R *is called semipotent ring [3], or* I_0- *ring in [2].*

Lemma 2.2. *[11, Lemma 1.2]. Let* A_R, B_R, C_R, D_R *be modules. Then:*

$[B,D]\circ J[A,B]\circ[C,A] \subseteq J[C,D]$.

Lemma 2.3. *Let* M_R, N_R *be modules such that* $\mathrm{Tot}\,[M,N] = J[M,N]$ *and* $a \in [M,N]$. *The following are equivalent:*

(1) $a \in J[M,N]$.

(2) $a\,b \in J(E_N)$ *for every* $b \in [N,M]$.

(3) $ma \in J(E_M)$ *for every* $m \in [N,M]$.

(4) $d\,a\,d \in J[N,M]$ *for every* $d \in [N,M]$.

Proof. It is clear that $(1) \Leftrightarrow (2)$ *and* $(1) \Leftrightarrow (3)$. $(1) \Rightarrow (4)$ *by lemma 2.2.*

$(4) \Rightarrow (1)$. *Suppose that* $a \notin J[M,N]$, *then* $a \notin \mathrm{Tot}\,[M,N]$. *Thus, there exists* $d \in [N,M]$ *such that* $0 \neq d = dad \in [N,M]$. *Since by assumption* $d \in J[N,M]$ *and* $0 \neq ad = (ad)^2 \in J(E_N)$ *a contradiction.*

Lemma 2.4. *Let* M_R, W_R *be modules such that* $\text{Tot}\,[M,W] = \nabla[M,W]$ *and let* $a \in [M,W]$. *The following are equivalent:*

(1) $Im(a) << W$.

(2) $a\,b \in \nabla(E_W)$ *for every* $b \in [W,M]$.

(3) $m\,a \in \nabla(E_M)$ *for every* $m \in [W,M]$.

(4) $d\,a\,d \in \nabla[W,M]$ *for every* $d \in [W,M]$.

Proof. $(1) \Rightarrow (2)$ *it is clear.* $(2) \Rightarrow (3)$. *Consider* $m\,a \notin \nabla(E_M)$ *for some* $m \in [W,M]$. *Then* $a \notin \nabla[M,W] = \text{Tot}\,[M,W]$, *so there exists* $g \in [W,M]$ *such that* $0 \neq a\,g = (a\,g)^2 \in E_W$. *By assumption* $a\,g \in \nabla(E_W)$, *so* $a\,g = 0$ *a contradiction.* $(3) \Rightarrow (4)$ *it is clear hence* $Im(dad) \subseteq Im(da)$ *for every* $d \in [W,M]$. $(4) \Rightarrow (1)$. *Suppose* $Im(a)$ *not small in* W, *then* $a \notin \nabla[M,W] = \text{Tot}\,[M,W]$, *so there exists* $d \in [W,M]$ *such that* $0 \neq d\,a = (d\,a)^2 \in E_M$, *by assumption* $d\,a \in \nabla(E_M)$, *so* $d\,a = 0$ *a contradiction.*

A module V_R *is said to be quasi-projective [10], if given an epimorphism* $l: V \to W$ *and any morphism* $a: V \to W$ *there exists* $m: V \to V$ *such that* $l\,m = a$. *Recall that a module* W_R *is locally projective [6], if for every submodule* $B \subseteq W$, *which is not small in* W *there exists a projective direct summand* $0 \neq P \subseteq^{\oplus} W$ *with* $P \subseteq B$. *F. Kasch in [6], proved that a module* W *is locally projective if and only if,* $\text{Tot}[M,W] = \nabla[M,W]$ *for all* $M \in \text{mod} - R$. *The next result gives new characterizes of this module.*

Proposition 2.5. *Let* W_R *be a module. The following conditions are equivalent:*

(1) *A module* W_R *is locally projective.*

(2) $\mathrm{Tot}[M,W]=\nabla[M,W]$ *for all* $M\in mod-R$.

(3) $\mathrm{Tot}[P,W]=\nabla[P,W]$ *for every quasi-projective module* $P\in mod-R$.

Proof. $(1)\Leftrightarrow(2)$. *By [6, Theorem 2.2].* $(2)\Rightarrow(3)$. *It is clear.* $(3)\Rightarrow(1)$. *Let* B *be a not small submodule of* W. *By [1, Theorem 2.3] denote by* $s:P\to B$ *a projection extension of* B *(i.e.,* P *can be free) and by* $t:B\to W$ *the inclusion. Then, for* $f=ts:P\to W$, $Im(f)=B$, *by assumption* $f\notin\nabla[P,W]$. *Since any projective module is quasi-projective, there exists* $g\in\nabla[W,P]$ *such that* $0\neq gf=(gf)^2\in E_P$. *Then by [6, Lemma 1.1], there exists* $0\neq P_0\subseteq^{\oplus}P$ *and* $0\neq B_0\subseteq^{\oplus}W$ *such that*

$$P_0\ni x\to f(x)\in B_0$$

is an isomorphism. Thus, B_0 *is a nonzero projective direct summand of W contained in B.*

Lemma 2.6. *Let* Q_R, N_R *be modules such that* $\mathrm{Tot}[Q,N]=\Delta[Q,N]$ *and* $a\in[Q,N]$. *The following are equivalent:*

(1) $Ker(a)\leq_e Q$.

(2) $m\,a\in\Delta(E_Q)$ *for every* $m\in[N,Q]$.

(3) $a\,b\in\Delta(E_N)$ *for every* $b\in[N,Q]$.

(4) $d\,a\,d\in\Delta[N,Q]$ *for every* $d\in[N,Q]$.

Proof. $(1)\Rightarrow(2)$ *it is clear.* $(2)\Rightarrow(3)$. *Consider* $a\,b\notin\Delta(E_N)$ *for some* $b\in[N,Q]$, *then* $Ker(a\,b)$ *is not large in N. Since* $Ker(a\,b)=b^{-1}(Ker(a))$, *follows* $a\notin\Delta[Q,N]=\mathrm{Tot}[Q,N]$. *So there exists* $g\in[N,Q]$ *such that* $0\neq g\,a=(ga)^2\in E_Q$, *hence by assumption* $g\,a\in\Delta(E_Q)$, *follows* $g\,a=0$ *a contradiction.* $(3)\Rightarrow(4)$.

We have $Ker\,(a\,d) \subseteq Ker\,(d\,a\,d)$ *for every* $d \in [N,Q]$. *By assumption* $a\,d \in \Delta(E_N)$, *so* $d\,a\,d \in \Delta[N,Q]$. $(4) \Rightarrow (1)$. *Suppose* $Ker\,(a)$ *is not large in Q, then* $a \notin \Delta[Q,N] = \mathrm{Tot}\,[Q,N]$, *so there exists* $d \in [N,Q]$ *such that* $d = d\,a\,d \notin \Delta[N,Q]$ *a contradiction.*

A module M_R *is called quasi-injective [8] if, for every submodule D of M and every* $f \in [D,M]$ *there exists* $g \in E_M$ *such that* $gt = f$, *where* $t: D \to M$ *the inclusion. Recall a module* Q_R *is locally injective [6] if, for every submodule* $B \subseteq Q$, *which is not large in Q, there exists an injective submodule* $0 \neq V \subseteq Q$, *with* $B \cap V = 0$. *F. Kasch in [6], proved that a module* Q *is locally injective if and* only if, $\mathrm{Tot}[Q,N] = \Delta\,[Q,N]$ *for all* $N \in mod - R$. *The next result gives new characterizes of this module:*

Theorem 2.7. *Let* Q_R *be a module. The following conditions are equivalent:*

(1) *A module* Q_R *is locally projective.*

(2) $\mathrm{Tot}[Q,N] = \Delta\,[Q,N]$ *for all* $N \in mod - R$.

(3) $\mathrm{Tot}[Q,U] = \Delta\,[Q,U]$ *for every quasi-injective module* $U \in mod - R$.

Proof. $(1) \Leftrightarrow (2)$ *by [6, Theorem 2.2].* $(2) \Rightarrow (3)$ *it is clear.* $(3) \Rightarrow (1)$. *Let* K *be a not large submodule of Q. By [1, Theorem 2.3] denote by* $n: Q \to Q/K$ *the natural epimorphism with* $Ker\,(n) = K$, *and let* $t: Q/K \to I$ *be a monomorphism into an injective module* I *(e.g.,* I *is the injective hull of* Q/K. *Then, for* $g = t\,n$, $Ker\,(g) = K$ *and by assumption* $g \notin \Delta\,[Q,I]$. *Since any injective module is quasi-injective, there exists* $y \in \Delta\,[I,Q]$ *such that* $0 \neq gy = (gy)^2 \in E_I$. *Then by [6, Lemma 1.1], there exists* $0 \neq I_0 \subseteq^{\oplus} I$ *and* $Q_0 \subseteq^{\oplus} Q$ *such* *that* $I_0 \ni x \to g(x) \in Q_0$ *is an isomorphism. Thus,* Q_0 *and* I_0 *are injective,* Q_0 *is a direct summand of* Q *and* $Q_0 \cap K = 0$.

Following [3], let M_R *be a module and* $K \subseteq^{\oplus} M$ *then* $K \subseteq J(M)$ *if and only if* $K \cap J(M) = J(K)$. *Putting* $\Gamma(M) = \{K : K \subseteq^{\oplus} M; J(K) = K\}$. *Not that for any projective module* P, $\Gamma(P) = \{0\}$. *In addition to, if* $J(M) << M$ *for some module* $M \in mod - R$ *then* $\Gamma(M) = \{0\}$. *A module* M_R *is called an* $I -$*module, if for every submodule K of M,* $K \not\subset J(M)$ *there exists a projective direct summand* $0 \neq B \subseteq^{\oplus} M$, *with* $B \subseteq K$.

Lemma 2.8. *Let* M_R, N_R *be modules such that* $\Gamma(M) = \Gamma(N) = \{0\}$, $\mathrm{Tot}[M,N] = I\,[M,N]$ *and let* $a \in I\,[M,N]$. *The following are equivalent:*

(1) $Im\,(a) \subseteq J(N)$.

(2) $a\, b \in I(E_N)$ *for every* $b \in [N,M]$.

(3) $m\, a \in I(E_M)$ *for every* $m \in [N,M]$.

(4) $da\,d \in I[N,M]$ *for every* $d \in [N,M]$.

Proof. $(1) \Rightarrow (2)$ *it is clear.* $(2) \Rightarrow (3)$. *Consider* $m\,a \notin I(E_M)$ *for some* $m \in [N,M]$. *Then* $a \notin I\,[M,N] = \mathrm{Tot}\,[M,N]$, *so there exists* $g \in [N,M]$ *such that* $0 \neq a\,g = (ag)^2 \in E_N$. *Thus by assumption* $a\,g \in I(E_N)$, *therefore* $Im\,(a\,g) \in \Gamma(N) = \{0\}$, *so* $a\,g = 0$ *a contradiction.* $(3) \Rightarrow (4)$. *We have by assumption* $da \in I(E_M)$ *for every* $d \in [N,M]$ *so* $Im(da) \subseteq J(M)$, *therefore* $Im(dad) \subseteq Im(d\,a) \subseteq J(M)$, *thus* $da\,d \in I[N,M]$. $(4) \Rightarrow (1)$. *Suppose* $Im\,(a) \not\subset J(N)$, *then* $a \notin I\,[M,N] = \mathrm{Tot}\,[M,N]$, *so there exists* $d \in [N,M]$ *such that* $d = dad = \notin I[N,M]$, *a contradiction with our assumption.*

Kasch and Mader in [7], started to study conditions on modules W, and Q for which imply that $\mathrm{Tot}[M,W]=\nabla[M,W]=J[M,W]$ *and* $\mathrm{Tot}[Q,N]=\Delta[Q,N]=\quad J[Q,N]$ *for every* $M,N\in mod-R$. *Next we characterize the right module* N_R *for which* $\mathrm{Tot}[M,N]=I[M,N]$ *for all* $M\in mod-R$.

Proposition 2.9. *Let* N_R *be a module and* $\Gamma(N)=\{0\}$. *The following are equivalent:*

(1) *A module N is an* $I-$*module.*

(2) $\mathrm{Tot}[M,N]=I[M,N]$ *for all* $M\in mod-R$.

(3) $\mathrm{Tot}[P,N]=I[P,N]$ *for every quasi-projective module* $P\in mod-R$.

Proof. $(1)\Rightarrow(2)$. *Let* $a\in\mathrm{Tot}[M,N]$, *suppose that* $a\notin I[M,N]$, *by assumption there exists a projective direct summand* $0\neq B\subseteq^{\oplus}N$, *with* $B\subseteq Im(a)$. *By the modular law* $B\subseteq^{\oplus}Im(a)$. *Denote by* $p:Im(a)\to B$ *the projection. Since* B *is projective, the epimorphism* $pa:M\to B$ *splits, hence* $M=D\oplus Ker(pa)$ *for some submodule D of M and* $D\ni x\to pa(x)\in B$ *is an isomorphism. By [6, Lemma 1.1 and 1.2] there exists* $b\in[N,M]$ *such that* $0\neq ba=(ba)^2\in E_N$ *and* $ba\in[N,M]a$, *a contradiction. Consider* $a\in I[M,N]$, *if* $a\notin\mathrm{Tot}[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\neq ba\quad=(ba)^2\in E_N$, *so* $Im(ba)\in\Gamma(N)=\{0\}$, *a contradiction, thus* $I[\mathrm{M},\mathrm{N}]\subseteq\mathrm{Tot}[M,N]$. $(2)\Rightarrow(3)$ *it is clear.* $(3)\Rightarrow(1)$. *Let* K *be a submodule of* N *with* $K\not\subset J(N)$. *By [1, Theorem 2.3], denote by* $d:P\to K$ *a projective extension of* K *(i.g.,* P *can be free) and by* $t:K\to N$ *the inclusion. Then, for* $g=td$, $Im(g)=K\not\subset J(N)$, *and* $g\notin I[P,N]$. *Since any projective module is quasi-projective, then by assumption there exists* $l:N\to P$ *such that* $0\neq lg=(lg)^2\in E_P$. *Then by [6, Lemma 1.1] there exists* $0\neq P_0\subseteq^{\oplus}P$ *and* $K_0\subseteq^{\oplus}N$ *such that* $P_0\ni x\to g(a)\in K_0$ *is an isomorphism. Thus,* K_O *is a nonzero projective direct summand of N contained in K.*

Lemma 2.10. *[7, Corollary 1.10]. For arbitrary modules* $M, W, X, Y \in mod-R$ *we have:*

$[W,Y] \circ \mathrm{Tot}[M,W] \circ [X,M] \subseteq \mathrm{Tot}[X,Y]$.

Theorem 2.11. *Let* M_R, N_R *be modules with* $\Gamma(N)=\{0\}$. *The following are equivalent:*

(1) $\mathrm{Tot}[M,N] = I[M,N]$.

(2) *For any* $a \in [M,N] \setminus I[M,N]$ *there exists* $0 \neq P \subseteq^{\oplus} M$ *such that* $Ker(a) \cap P = 0$.

Proof. $(1) \Rightarrow (2)$. *Let* $a \in [M,N] \setminus I[M,N]$ *then* $a \notin \mathrm{Tot}[M,N]$, *so there exists* $b \in [N,M]$ *such that* $0 \neq ba = (ba)^2 \in E_M$. *By [6, Lemma 1.1] there exist direct summands* $0 \neq A \subseteq^{\oplus} M$, $B \subseteq^{\oplus} N$, *such that the mapping* $A \ni a \to a(a) \in B$ *is an isomorphism, so* $Ker(a) \cap B = 0$. $(2) \Rightarrow (1)$. *It is clear that* $I[M,N] \subseteq \mathrm{Tot}[M,N]$, *hence* $\Gamma(N)=\{0\}$. *Let* $a \in [M,N] \setminus I[M,N]$, *by our hypothesis there exists* $0 \neq P \subseteq^{\oplus} M$ *such that* $Ker(a) \cap P = 0$. *Let* $p: M \to P$ *be the projection and let* $b \in [a(P), M]$ *given by* $b\,a(x) = x$ *for all* $x \in P$, *then* $b\,a\,p \in E_M$. *In addition to, for every* $x \in P$, $b\,a\,p(x) = b\,a(x) = x$, *therefore* $b\,a\,p = I_{a(P)}$. *Since* $b\,a\,p = I_{a(P)} \notin \mathrm{Tot}(E_{a(P)})$, *follows (by lemma 2.10, for* $X = Y$*) that* $a \notin \mathrm{Tot}[M,N]$. *So,* $\mathrm{Tot}[M,N] \subseteq I[M,N]$.

A module M_R *is called* $I-$*module [4], if for every submodule K of M such that* $K \not\subset J(M)$ *there exists a projective direct summand* $0 \neq W \subseteq^{\oplus} M$ *with* $W \subseteq K$. *Recall that a ring R is a semipotent ring [3], if every principal right ideal not contained in* $J(R)$ *contains a nonzero idempotent. Next we characterize the right module* N_R *for* *which* $\mathrm{Tot}[M,N]=I\,[M,N]=J[M,N]$ *for all* $M\in mod-R$. *We need the following theorem:*

Theorem 2.12. *[4,Theorem 2.10]. Let* N_R *be a module with* $\Gamma(N)=\{0\}$. *The following conditions are equivalent:*

(1) *The module* N_R *is an* $I-$*module.*

(2) $[M,N]$ *is an* $I-$*semipotent for all* $M\in mod-R$.

(3) $\mathrm{Tot}[M,N]=I\,[M,N]$ *for all* $M\in mod-R$.

Proposition 2.13. *Let* N_R *be a module with* $\Gamma(N)=\{0\}$. *The following conditions are equivalent:*

(1) $\mathrm{Tot}[M,N]=I\,[M,N]=J[M,N]$ *for all* $M\in mod-R$.

(2) N_R *is an* $I-$*module with* E_N *semipotent.*

(3) N_R *is an* $I-$*module such that every* $d\in E_N$ *with* $Im(1-d)\subseteq J(N)$ *is one-to-one.*

Proof. $(1)\Rightarrow(2)$. *Since* $\mathrm{Tot}[M,N]=I\,[M,N]$ *for all* $M\in mod-R$ *so by theorem 2.12,* N_R *is an* $I-$*module. On the other hand, since* $\mathrm{Tot}[M,N]=J[M,N]$ *for all* $M\in mod-R$ *then by [13, Theorem 4.1],* E_N *is semipotent.* $(2)\Rightarrow(3)$. *Let* $b\in E_N$ *with* $Im(1-b)\subseteq J(N)$, *then* $1-b\in I(E_N)=J(E_N)$ *by assumption. Hence* b *is one-to-one.* $(3)\Rightarrow(1)$. *Since* N_R *is an* $I-$*module, then by theorem 2.12,* $\mathrm{Tot}[M,N]=I\,[M,N]$ *for all* $M\in mod-R$. *On the other hand, it is clear that* $J[\mathrm{M,N}]\subseteq\mathrm{Tot}[M,N]$. *Let* $a\in\mathrm{Tot}[M,N]$ *then* $a\in I[M,N]$, *so* $Im(a)\subseteq J(N)$ *and for all* $b\in[N,M]$, $Im(ab)\subseteq Im(a)\subseteq J(N)$, *thus* $Im(1-(1-ab))=Im(ab)\subseteq J(N)$ *and* $ab\in E_N$ *by assumption* $1-ab$ *is one-to-one for all* $b\in[N,M]$, *therefore* $a\in J[M,N]$.

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