# On (I-) Semipotent [M,N]

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## ABSTRACT

**Let $M_R$ and $N_R$ be modules, we use $[M,N] = hom_R(M,N)$, so $[M,N]$ is an $(E_M, E_N)-$bimodule. Some of the interesting questions are when the total equals the Jacobson radical, $\Delta E_M$, $\nabla E_M$ and $I(E_M)$ for some module $M_R$. In this paper we study the question is when the total equals the ideal $I(E_M)$. New results obtained include: (1) A projective module $P_R$ is an $I_0-$module if and only if $E_P = End_R(P)$ is an $I-$semipotent. (2) A ring $R$ is a semipotent ring if and only if, endomorphism ring of any projective module $P_R$ is an $I-$semipotent. (3) For any projective module $P_R$; $Tot(E_P) = I(E_P)$ if and only if, $P$ is an $I_0-$ module. (4) For any ring $R$; $Tot(R) = J(R)$ if and only if, $[M,P]$ is an $I-$semipotent for any projective module $P_R$ and any module $M_R$ which equivalent that, $Tot[M,P] = I[M,P]$ for any projective module $P_R$ and any module $M_R$, which also, equivalent $[M,P]$ is semipotent for any finitely generated projetive module $P_R$ and any module $M_R$.**

**Key Words:** *(I-) Semipotent Rings, $I_0-$Rings, $I_0-$modules, The total, Jacobson radical, (co) singular ideal, Endomorphism rings, $hom_R(M,N)$.*

# حـــــول المودولات $[M,N]$ الـــ $I$ –شبــــه جـــامــــدة

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## المـــلخص

**ليكن $M_R$ و $N_R$ مودولين فوق الحلقة $R$ ولنفرض أن $[M,N] = hom_R(M,N)$ مجموعة جميع التشاكلات المودولية بين المـــودولين $M$ و $N$. فنحـــصل بـــذلك علـــى مـــودول ثنـــائي فـــوق الحلقات $E_M$ و $E_N$.**

**إن الهدف من هذا البحث هو دراسة البنى الجزئية للمودول الثنائي $[M,N]$ ونذكر منهـــا الأســـاس والمثالي المنفرد والمثالي المنفرد الثنوي والتوتـــال بـــشكل عـــام و $I\,[M,N]$ وعلاقتـــه بالتوتـــال $Tot\,[M,N]$ بشكل خاص. وقد تم التوصل من خلال هذه الدراسة إلى نتائج جديدة منها:**

1. **المودول الإسقاطي $P$ هو $I_0-$مودول عندما وفقط عندما تكون $E_P = End_R(P)$ عبارة عن حلقة $I-$ شبه جامدة.**
2. **الحلقة $R$ هي حلقة شبه جامدة عندما وفقط عندما تكون حلقة التـــشاكلات $E_P$ لأجـــل أي مـــودول إسقاطي $P$ عبارة عن حلقة $I-$شبه جامدة.**
3. **لأجل أي مودول إســـقاطي $P_R$ فـــإن المـــودول $P$ هـــو $I_0-$مـــودول عنـــدما وفقـــط عنـــدما $Tot\,(E_P) = I\,(E_P)$.**
4. **لأجل أي حلقة $R$ فإن $Tot\,(R) = J(R)$ عندما وفقط عندما المـــودول $[M,P]$ هـــو $I-$ شـــبه جامـــد وذلـــك لأجـــل أي مـــودول إســـقاطي $P \in mod-R$ ولأجـــل أي مـــودول $M \in mod-R$ والذي يكافئ بدوره أن $Tot\,[M,P] = I\,[M,P]$ وذلك لأجل أي مـــودول إسقاطي $P \in mod-R$ ولأجل أي مودول $M \in mod-R$ والذي أيضاً بدوره يكافئ المودول $[M,P]$ شبه جامد وذلك لأجل أي مودول إسقاطي منتـــه التوليـــد $P \in mod-R$ ولأجـــل أي مودول $M \in mod-R$.**

**الكلمات المفتاحية:** الحلقة شبه الجامدة، $I_0-$حلقـــة، $I_0-$مـــودول، التوتـــال، أســـاس جاكبسون، المثالي المنفرد، حلقة التشاكلات لمودول، $hom_R(M,N)$.

# Introduction.

In this paper rings $R$ are associative with identity unless otherwise indicated. All modules over a ring $R$ unitary right modules. A submodule $N$ of a module $M$ is said to be small in $M$ if $N+K \neq M$ for any proper submodule $K$ of $M$ ,[4]. A submodule $N$ of a module $M$ is said to be large (essential) in $M$ if $N \cap K \neq 0$ for any nonzero submodule $K$ of $M$ [4]. If $M$ is an $R-$module, the radical of $M$ denoted by $J(M)$ is defined to be the intersection of all maximal submodules of $M$ . Also, $J(M)$ coincides with the sum of all small submodules of $M$ . It my happen that $M$ has no maximal submodules in which case $J(M)=M$ [8]. Thus, for a ring $R$ , $J(R)$ is the Jacobson radical of $R$ . For a submodule $N$ of a module $M$ , we use $N \subseteq^{\oplus} M$ to mean that $N$ is a direct summand of $M$ , and we write $N \leq_e M$ and $N << M$ to indicate that $N$ is a large, respectively small, submodule of $M$ . If $M_R$ is a module, we use the notation $E_M = End_R(M)$ and we write $\Delta E_M = \{\alpha : \alpha \in E_M ; Ker(\alpha) \leq_e M\}$ , $\nabla E_M = \{\alpha : \alpha \in E_M ; Im(\alpha) << M\}$ and $I(E_M) = \{\alpha : \alpha \in E_M ; Im(\alpha) \subseteq J(M)\}$ . It is well known that $\Delta E_M$ , $\nabla E_M$ and $I(E_M)$ are ideals in $E_M$ [4]. If $M_R$ and $N_R$ are modules, we use $[M,N] = hom_R(M,N)$ . Thus $[M,N]$ is an $(E_M, E_N)-$bimodule. Our main concern is about the substructures of $hom_R(M,N)$ and the semipotent of $hom_R(M,N)$ (see [9]).

The total is a concept that was first introduced by Kasch [2] in 1982. In the study of the total, some of the interesting questions are when the total equals the Jacobson radical, $\Delta E_M$ , $\nabla E_M$ and $I(E_M)$ . In this paper we study the question is when the total equals the ideal $I(E_M)$ . In section 2, it is proved that, for any two modules $M_R$ , $N_R$ , $[M,N]$ is an $I-$semipotent if and only if, for any $\alpha \in [M,N] \setminus I[M,N]$ there exists $\gamma \in [N,M]$ such that $\gamma\alpha\gamma = \gamma \notin I[N,M]$ . Also, proved that if $\Gamma(M)=\{0\}$ ( or, $\Gamma(N)=\{0\}$ ) then $Tot\ [M,N] = I[M,N]$ if and only if, $[M,N]$ is $I-$semipotent. The main result states that a ring $R$ is a semipotent ring if and only if, $Tot\ [M,P] = I[M,P]$ for any module

$M \in mod-R$ and any projective module $P \in mod-R$. Basic properties on $I-$semipotentness of *[M,N]* are proved in this section.

## (I-) SEMIPOTENT

Recall a ring $R$ is semipotent [5,3] if each one-sided ideal not contained in $J(R)$ contains a nonzero idempotent. The semipotent rings generalize as following:

**Lemma 2.1.** [7, Lemma 19]. The following conditions are equivalent for an ideal $I$ of a ring $R$:

(1) If $T \not\subset I$ is a right (resp. left) ideal there exists $e^2 = e \in T \setminus I$.

(2) If $a \notin I$ there exists $e^2 = e \in aR \setminus I$ (resp. $e^2 = e \in Ra \setminus I$).

(3) If $a \notin I$ there exists $x \in R$ such that $x = xax \notin I$.

Let $R$ be a ring and $I$ is an ideal of $R$, recall $R$ is an $I-$semipotent [7], if the conditions in lemma 2.1, are satisfied. If $I = J(R)$ then $R$ is semipotent if and only if $R$ is $I-$ semipotent.

**Corollary 2.2.** Let $I$ be an ideal of a ring $R$. If $R$ is $I-$semipotent then $J(R) \subseteq I$.

Proof. Suppose $J(R) \not\subset I$ there exists $a \in J(R)$, $a \notin I$, so $x = xax \notin I$ for some $x \in R$. Since $x \neq 0$ then $0 \neq (ax)^2 = ax \in J(R)$ this is a contradiction.

Let $M_R$ be a module, letting

$$I = I(E_M) = \{\ \alpha : \alpha \in E_M ; Im(\alpha) \subseteq J(M)\}$$

It is clear that $I = I(E_M)$ is an ideal in $E_M$ and

$$I(E_M) = \{\ \alpha : \alpha \in E_M ; \beta\alpha \in I(E_M), \text{for all}\ \beta \in E_M\}$$
$$I(E_M) = \{\ \alpha : \alpha \in E_M ; \alpha\beta \in I(E_M), \text{for all}\ \beta \in E_M\}$$

Recall a projective module $P$ is an $I_0-$module if, any submodule $A$ of $P$, $A \not\subset J(P)$ contains a nonzero direct summand of $P$.

**Theorem 2.3.** [3, Theorem 3.2]. Any projective module over semipotent ring is an $I_0-$module.

**Theorem 2.4.** Let $P$ be a projective module. The following are equivalent:
(1) $P$ is an $I_0$ −module.
(2)For any $\alpha \in E_P, \alpha \notin I = I(E_P)$ there exists $0 \neq B \subseteq^{\oplus} P$, $B \subseteq Im(\alpha)$.
(3) For any $\alpha \in E_P$, $\alpha \notin I$, $0 \neq Im(\beta\alpha) \subseteq^{\oplus} P$ for some $\beta \in E_P$.
(4) For any $\alpha \in E_P$, $\alpha \notin I$, $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$ for some $\beta \in E_P$.
(5) $E_P$ is an $I-$ semipotent ring.

Proof. We will proceed by showing that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. It is clear. $(2) \Rightarrow (3)$. Let $\alpha \in E_P \setminus I$, then $Im(\alpha) \not\subset J(P)$ so there exists $0 \neq K \subseteq^{\oplus} P$, $K \subseteq Im(\alpha)$. Let $\beta$ be the projection of $P$ onto $K$ then $0 \neq \beta^2 = \beta \in E_P$ and $Im(\beta) \subseteq Im(\alpha)$ therefore $Im(\beta\alpha) \subseteq Im(\beta) = K \subseteq^{\oplus} P$. So (3) holds.

$(3) \Rightarrow (5)$. Let $\alpha \in E_P \setminus I$, then $0 \neq Im(\beta\alpha) \subseteq^{\oplus} P$ for some $\beta \in E_P$. Since $Im(\beta\alpha)$ is projective then $Ker(\beta\alpha) \subseteq^{\oplus} P$, by [8Lemma 3.1] there exists $0 \neq \mu \in E_P$, $(\beta\alpha)\mu(\beta\alpha) = \beta\alpha$. Thus $0 \neq (\mu\beta\alpha)^2 = \mu\beta\alpha \in (E_P).\alpha$ by Lemma 2.1 (2), $E_P$ is $I-$ semipotent.

$(5) \Rightarrow (4)$. Let $\alpha \in E_P \setminus I$, then there exists $\beta \in E_P$ such that $\beta = \beta\alpha\beta \notin I$. Since $0 \neq (\beta\alpha)^2 = (\beta\alpha) \in E_P$ then $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$.

$(4) \Rightarrow (1)$. Let $A$ be a submodule of $P$, $A \not\subset J(P)$ then there exists a maximal submodule $D$ of $P$, $A \not\subset D$ therefore $P = A + D$ by [1, Lemma 2.2] there exists $\alpha, \beta \in E_P$, $Im(\alpha) \subseteq A$, $Im(\beta) \subseteq D$ and $1 = \alpha + \beta$,furthermore $\alpha \notin I$ ,if $\alpha \in I$ then $P = Im(\alpha) + Im(\beta) \subseteq J(P) + D \subseteq D \subseteq P$ thus $P = D$ this is a contradiction, by our assumption there exists $\beta \in E_P$, $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$ and $Im(\alpha\beta) \subseteq Im(\alpha) \subseteq A$. Thus $P$ is an $I_0-$ module. Our proof is completed.

**Theorem 2.5.** For any ring $R$ the following conditions are equivalent:
(1) $R$ is a semipotent ring.
(2) Each projective module $P \in mod - R$ is an $I_0$ −module.

(3) $E_P$ is an $I-$ semipotent ring for any projective module $P \in mod-R$, where $I = I(E_P)$.

Proof. $(1) \Rightarrow (2)$. By Theorem 2.3. $(2) \Rightarrow (3)$. By Theorem 2.4. $(3) \Rightarrow (1)$ .Follows

from the fact that $R_R$ is a projective module and $J(R) = I = I(R_R)$.

Let $M_R$, $N_R$ be modules. We put

$$I[M,N] = \{ \alpha : \alpha \in [M,N] ; Im(\alpha) \subseteq J(N) \}$$

It is clear that

$$I[M,N] \subseteq \{ \alpha : \alpha \in [M,N] ; \beta\alpha \in I(E_M) \text{ for all } \beta \in [M,N] \}$$

$$I[M,N] \subseteq \{ \alpha : \alpha \in [M,N] ; \beta\alpha \in I(E_N) \text{ for all } \beta \in [M,N] \}$$

Since any small submodule of $N$ contained in $J(N)$ then $\nabla[M,N] \subseteq I[M,N]$. If $J(N) << N$ then $\nabla[M,N] = I[M,N]$. Thus, $I = I(E_M) = I[M,M] = \{\alpha : \alpha \in E_M ; Im(\alpha) \subseteq J(M)\}$. In particular, for a ring $R$, $I(R) = I[R,R] = J[R,R] = J(R)$.

**Lemma 2.6.** Let $M_R$, $N_R$ be modules. The following conditions are equivalent:

(1)If $\alpha \in [M,N] \setminus I[M,N]$, there exists $\beta \in [N,M] ; 0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$, $\beta\alpha \notin I(E_M)$.

(2)If $\alpha \in [M,N] \setminus I[M,N]$, there exists $\beta \in [N,M] ; 0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$, $\alpha\beta \notin I(E_N)$.

(3)If $\alpha \in [M,N] \setminus I[M,N]$, there exists $\gamma \in [N,M]$ ; $\gamma\alpha\gamma = \gamma \notin I[N,M]$.

Proof. Suppose (1) holds. Then $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ and $\beta\alpha \notin I(E_M)$ for some $\beta \in [N,M]$. By letting $\gamma = \beta\alpha\beta \in [N,M]$ we have $\gamma\alpha\gamma = \gamma \neq 0$ and $\gamma \notin I[N,M]$ because $\beta\alpha \notin I(E_M)$, giving (3). Suppose (3) holds. Then $0 \neq \gamma\alpha = (\alpha\gamma)^2 \in E_M$ and $\gamma\alpha \notin I(E_M)$ because $\gamma \notin I[N,M]$ gives (1). Similarly, the equivalence $(2) \Leftrightarrow (3)$ holds.

We call that $[M,N]$ is $I-$semipotent if, the conditions in lemma 2.6 are satisfied. If $I[M,N] = J[M,N]$ then $[M,N]$ is semipotent if and only if $[M,N]$ is $I-$ semipotent.

**Lemma 2.7.** Let $M_R$, $N_R$ be modules. If $[M,N]$ is $I-$semipotent then $J[M,N] \subseteq I[M,N]$. In particular, if $E_M$ is an $I-$semipotent ring then $J(E_M) \subseteq I(E_M)$.

Proof. Let $\alpha \in J[M,N]$. Suppose that $\alpha \notin I[M,N]$ there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha\beta = \beta \notin I[N,M]$. Since $\alpha \in J[M,N]$, then $0 \neq \alpha\beta = (\alpha\beta)^2 \in J(E_N)$ this is a contradiction.

**Proposition 2.8.** Let $M_R$, $N_R$ be modules. If $[M,N]$ is an $I-$semipotent then the following hold:
(1) $I[M,N] = \{ \alpha : \alpha \in [M,N];\ \beta\alpha \in I(E_M) \text{ for all } \beta \in [N,M] \}$.
(2) $I[M,N] = \{ \alpha : \alpha \in [M,N];\ \alpha\beta \in I(E_N) \text{ for all } \beta \in [N,M] \}$.

Proof. Suppose $[M,N]$ is an $I-$ semipotent.

(1) It is clear that $I[M,N] \subseteq \{ \alpha : \alpha \in [M,N];\ \beta\alpha \in I(E_M) \text{ for all } \beta \in [N,M] \}$. Let $\alpha \in [M,N]$ such that $\beta\alpha \in I(E_M)$ for all $\beta \in [N,M]$ and Suppose that $\alpha \notin I[M,N]$ then there exists $0 \neq \gamma \in [N,M]$, such that $\gamma\alpha\gamma = \gamma \notin I[N,M]$. Thus, $Im(\gamma) \subseteq Im(\gamma\alpha) \subseteq J(M)$ so $\gamma \in I[N,M]$ this is a contradiction, therefore $\alpha \in I[M,N]$. Proof (2) is analogous.

Let $M_R$ be a module and $K \subseteq^{\oplus} M$, then $K \subseteq J(M)$ if and only if $J(K) = K \cap J(M) = K$. Putting

$$\Gamma(M) = \{ K : K \subseteq^{\oplus} M;\ J(K) = K \}$$

Not that for any projective module $P$, $\Gamma(P) = \{0\}$. In addition to, if $J(M) << M$ for some module $M \in mod-R$ then $\Gamma(M) = \{0\}$.

Following [7], Let $M_R$, $N_R$ be modules. The total.

$$Tot[M,N] = \{ \alpha : \alpha \in [M,N];\ [N,M]\alpha \text{ contains no nonzero idempotent} \}$$
$$Tot[M,N] = \{ \alpha : \alpha \in [M,N];\ \alpha[N,M] \text{ contains no nonzero idempotent} \}$$

**Lemma 2.9.** Let $M_R$, $N_R$ be modules then the following hold:

(1) If $\Gamma(M) = \{0\}$ then for any $\alpha \in I[M,N]$; $[N,M]\alpha$ and $\alpha[N,M]$ contains no nonzero idempotents.

(2) If $\Gamma(N) = \{0\}$ then for any $\alpha \in I[M,N]$; $\alpha[N,M]$ and $[N,M]\alpha$ contains no nonzero idempotents.

Proof. (1). Suppose $\Gamma(M)=\{0\}$. Let $\alpha \in I[M,N]$ and suppose there exists $\beta \in [N,M]$ such that $(\beta\alpha)^2=\beta\alpha \in E_M$ then $Im(\beta\alpha) \subseteq^{\oplus} M$ and $Im(\beta\alpha) \subseteq \beta(J(N)) \subseteq J(M)$ so $Im(\beta\alpha) \in \Gamma(M)=\{0\}$ and $Im(\beta\alpha)=0$ thus $\beta\alpha=0$. Since *Tot [M, N]* contains no nonzero idempotents then $\alpha \in Tot\,[M,N]$ so $\alpha\,[N,M]$ contains no nonzero idempotents. Similarly (2) holds.

**Corollary 2.10.** The following hold:

(1) Let $M_R$ be a module with $\Gamma(M)=\{0\}$ then $I[M,N] \subseteq Tot\,[M,N]$ for all $N \in mod-R$.

(2) Let $N_R$ be a module with $\Gamma(N)=\{0\}$ then $I[M,N] \subseteq Tot[M,N]$ for all $M \in mod-R$.

In particular, if $M_R$ be a module and $\Gamma(M)=\{0\}$ then $I(E_M) \subseteq Tot(E_M)$.

Proof follows immediately from lemma 2.9.

**Theorem 2.11.** Let $M_R$, $N_R$ be modules then the following hold:

(1) If $\Gamma(M)=\{0\}$ then $Tot\,[M,N] = I[M,N]$ if and only if *[M,N]* is $I-$ semipotent.

(2) If $\Gamma(N)=\{0\}$ then $Tot\,[M,N] = I[M,N]$ if and only if *[M,N]* is $I-$ semipotent.

In particular, if $\Gamma(M)=\{0\}$ then $Tot\,[M,M] = I(E_M)$ if and only if $E_M$ is an $I-$ semipotent ring.

Proof.(1). Suppose $\Gamma(M)=\{0\}$.

$(\Rightarrow)$. Let $\alpha \in [M,N] \setminus I[M,N]$ then $\alpha \notin Tot\,[M,N]$. So $0 \neq (\beta\alpha)^2=\beta\alpha \in E_M$ for some $\beta \in [N,M]$ and $\beta\alpha \notin I(E_M)$ because $\Gamma(M)=\{0\}$. This shows that *[M,N]* is $I-$ semipotent.

$(\Leftarrow)$. We have by corollary 2.10(1), $I[M,N] \subseteq Tot\,[M,N]$. Let $\alpha \in Tot\,[M,N]$ and suppose $\alpha \notin I[M,N]$. So, for any $\beta \in [N,M]$, either $\alpha\beta=0$ or $\alpha\beta \neq (\alpha\beta)^2$. Hence *[M,N]* is not $I-$ semipotent. Similarly (2) holds.

**Corollary 2.12.** Let $P$ be a projective module. The following are equivalent:

(1) $P$ is an $I_0$ −module.

(2) $E_P$ is an $I$ −semipotent ring.

(3) $Tot(E_P) = I(E_P)$.

Proof. $(1) \Leftrightarrow (2)$. By Theorem 2.4. $(2) \Leftrightarrow (3)$. By Theorem 2.11, because $\Gamma(P) = \{0\}$.

**Proposition 2.13.** Let $P$ be a projective module. The following are equivalent:

(1) $E_P$ is an $I$ − semipotent ring.

(2) $[M,P]$ is $I$ − semipotent for all module $M \in mod - R$.

Proof. Suppose (1) holds. Let $\alpha \in [M,P] \setminus I[M,P]$ then $Im(\alpha) \not\subset J(P)$ by corollary 2.12, there exists $0 \neq K \subseteq^{\oplus} P$, $K \subseteq Im(\alpha)$. Let $\mu$ be the projection of $P$ onto $K$ then $0 \neq \mu^2 = \mu \in E_P$ and $Im(\mu) = Im(\mu\alpha) = K$. Since $P$ is projective there exists $\beta \in [P,M]$ such that $\mu\alpha\beta = \mu$ therefore $\beta\mu\alpha\beta = \beta\mu \in [P,M]$. Since for any $x \in P$, $\mu(x) \in P$ then $\beta\mu\alpha\beta\mu(x) = \beta\mu(x) \in [P,M]$ thus $\beta\mu\alpha\beta\mu = \beta\mu \in [P,M]$ and $0 \neq \beta\mu \notin I[P,M]$. By Lemma 2.6, $[M,P]$ is $I$ − semipotent, so (2) holds. Suppose (2) holds. Then $[P,P] = E_P$ is $I$ − semipotent.

**Theorem 2.14.** The following conditions are equivalent for any ring $R$:

(1) $R$ is a semipotent ring.

(2) $Tot\ (R) = J(R)$.

(3) $E_P$ is an $I$ −semipotent ring for any projective module $P \in mod - R$.

(4) $Tot(E_P) = I(E_P)$ for any projective module $P \in mod - R$.

(5) $[M,P]$ is $I$ −semipotent for any projective module $P \in mod - R$ and any module $M \in mod - R$.

(6) $Tot\ [M,P] = I[M,P]$ for any projective module $P \in mod-R$ and any module $M \in mod-R$.

(7) $E_P$ is a semipotent ring for any finitely generated projective module $P \in mod-R$.

(8) $[M,P]$ is semipotent for any finitely generated projective module $P \in mod-R$ and any module $M \in mod-R$.

Proof. $(1) \Leftrightarrow (2)$. By [11, Theorem 2.2]. $(1) \Leftrightarrow (3)$. By Theorem 2.5. $(3) \Leftrightarrow (4)$. By Corollary 2.12. $(3) \Leftrightarrow (5)$. By Proposition 2.13. $(5) \Leftrightarrow (6)$. By Theorem 2.11, since for any projective module $P$, $\Gamma(P) = \{0\}$ and for any module $M$. $(3) \Rightarrow (7)$. Obvious. $(7) \Rightarrow (3)$. Since $J(P) << P$ then $Tot(E_P) = J(E_P) = I(E_P)$. So (3) holds. $(7) \Leftrightarrow (8)$. By proposition 2.13, because $J(P) << P$.

**Corollary 2.15.** Let $P$ be an $I_0$ –projective module then for any module $M \in mod-R$; $I[M,P]=\{\alpha : \alpha \in [M,P]; \alpha\beta \in I(E_p) \text{ for all } \beta \in [P,M]\}$.

Proof. Since $P$ is an $I_0$ –projective then by corollary 2.12, $E_P$ is an $I-$ semipotent ring and by proposition 2.13, $[M,P]$ is an $I-$ semipotent for all $M \in mod-R$. Since $\Gamma(P) = \{0\}$ then by proposition 2.8(2), $I[M,P] = \{\alpha : \alpha \in [M,P];\ \alpha\beta \in I(E_p) \text{ for all } \beta \in [P,M]\}$.

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