The Total Of Rings and Modules

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ABSTRACT

Let M_R and N_R be two modules over associative ring R. The object of this paper is the study of substructures of $hom_R(M,N)$ such as, radical, the singular and co-singular ideal and the total. New results obtained include:

(a) For any modules M_R, N_R ; $[M,N]$ is regular if and only if, for any $\alpha \in [M,N]$, $Im(\alpha) \subseteq^{\oplus} N$ and $Ker(\alpha) \subseteq^{\oplus} M$.

(b) $[M,N]$ is semipotent if and only if, for any $\alpha \in [M,N]$, $\alpha \notin J[M,N]$ there exists $\beta \in [N,M]$ such that $Im(\alpha\beta) \subseteq^{\oplus} N$ and $Ker(\alpha\beta) \subseteq^{\oplus} N$.

(c) For any injective module Q_R and any projective module P_R with E_P semipotent, where $E_P = End_R(P)$; $\text{Tot}[Q,P] = J[Q,P] = \Delta[Q,P] = \nabla[Q,P]$.

(d) A ring R is semipotent with $J(R)$ is left $T-$ nilpotent if and only if, $[M,P]$ is semipotent which equivalent, $[P,M]$ is semipotent, for any projective module $P \in mod-R$, and any module $M \in mod-R$.

Key Words: (I-) Semipotent Rings, I_0 −Rings, I_0 −modules, The total, Jacobson radical, (co) singular ideal, Endomorphism rings, $hom_R(M,N)$.

تـــوتـــال الحـــلقـــات والمـــودولات

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المـــلخص

ليكن M_R و N_R مودولين فوق الحلقة R ولنفرض أن $[M,N] = hom_R(M,N)$ مجموعة جميع التشاكلات المودولية من M إلى N . فنحصل بذلك على مودول ثنائي فوق الحلقات E_M و E_N .

إن الهدف من هذا البحث هو دراسة البنى الجزئية للمودول الثنائي $[M,N]$ منها الجـذر والمثـالي المنفرد والمثالي المنفرد الثنوي والتوتال، وقد تم التوصل (من خلال هذه الدراسة) إلى نتائج جديدة منها:

1. **لأجل أي مودولين M و N فوق الحلقة R فإن $[M,N]$ منتظم عندما وفقط عنـدما لأجـل كـل عنصر $\alpha \in [M,N]$ فإن $Im(\alpha)$ حد مباشر في N و $Ker(\alpha)$ حد مباشر في M .**
2. **المـــودول الثنـــائي $[M,N]$ شـــبه جامـــد عنـــدما وفقـــط عنـــدما لأجـــل كـــل عنـــصر $\alpha \in [M,N] \setminus J[M,N]$ يوجـــد $\beta \in [N,M]$ بحيـــث يكـــون $Im(\alpha\beta)$ و $Ker(\alpha\beta)$ حدوداً مباشرة في المودول N .**
3. **لأجل أي مودول إدخال Q_R وأي مودول إسقاطي P_R يحقق $End_R(P)$ حلقة شبه جامدة فإن $Tot[Q,P] = J[Q,P] = \Delta[Q,P] = \nabla[Q,P]$.**
4. **الحلقة R شبه جامدة والجذر $J(R)$ عبارة عن $T-$مثالي عديم القوى من اليسار عنـدما وفقـط عندما يكون المودول $[M,P]$ شبه جامد، وهذا بدوره يكافئ أن المـودول $[P,M]$ شـبه جامـد وذلك لأجل أي مودول إسقاطي $P \in mod-R$ ولأجل أي مودول $M \in mod-R$.**

الكلمات المفتاحية: الحلقة شبه الجامدة، I_0-حلقة، I_0-مـودول، التوتـال، جـذر جاكبـسون، المثـالي المنفـرد، حلقـة التـشاكلات لمـودول، $hom_R(M,N)$.

1. Introduction

In this paper rings R are associative with identity unless otherwise indicated. All modules over a ring R unitary right modules. A submodule N of a module M is said to be small in M if $N+K\neq M$ for any proper submodule K of M , [3]. A submodule N of a module M is said to be large (essential) in M if $N\cap K\neq\{0\}$ for any nonzero submodule K of M [3]. If M is an $R-$module, the radical of M denoted by $J(M)$ is defined to be the intersection of all maximal submodules of M. Also, $J(M)$ coincides with the sum of all small submodules of M. It my happen that M has no maximal submodules in which case $J(M)=M$ [7]. Thus, for a ring R, $J(R)$ is the Jacobson radical of R. For a submodule N of a module M, we use $N\subseteq^{\oplus}M$ to mean that N is a direct summand of M, and we write $N\leq_e M$ and $N<<M$ to indicate that N is a large, respectively small, submodule of M. If M_R is a module, we use the notation $E_M=End_R(M)$ and we write $\Delta E_M=\{\alpha:\alpha\in E_M; Ker(\alpha)\leq_e M\}$, $\nabla E_M=\{\alpha:\alpha\in E_M; Im(\alpha)<<M\}$ and $I(E_M)=\{\alpha:\alpha\in E_M; Im(\alpha)\subseteq J(M)\}$. It is will known that $\Delta E_M, \nabla E_M$ and $I(E_M)$ are ideals in E_M [5]. If M_R and N_R are modules, we use $[M,N]=hom_R\ (M,N)$. Thus $[M,N]$ is an $(E_M,E_N)-$ bimodule. Our main concern is about the substructures of $hom_R(M,N)$ and the semipotent of $hom_R(M,N)$ (see [8]).

The total is a concept that was first introduced by Kasch in 1982. In the study of the total, some of the interesting questions are when the total equals the Jacobson radical, $\Delta E_M, \nabla E_M$ and $I(E_M)$. In section 2, it is proved that over quasi-frobenius ring $J(E_M)=\Delta E_M=\nabla E_M$ for any projective module (injective) module M. In section 3, it is proved that $\mathrm{Tot}[Q,P]=J[Q,P]=\Delta[Q,P]=\nabla[Q,P]$ for any injective module Q and any projective module P with E_P is semipotent. And for any projective module P with E_P is semipotent; $\mathrm{Tot}[M,P]=J[M,P]=\nabla[M,P]$ for all $M\in mod-R$. Also, proved that R is semipotent with $J(R)$ is left $T-$nilpotent if and only if $\mathrm{Tot}[M,P]=J[M,P]$ for any projective module $P \in mod - R$ and any module $M \in mod - R$ which equivalent $\mathrm{Tot}[P,M] = J[P,M]$ for any projective module $P \in mod - R$ and any $M \in mod - R$. Basic properties on semipotentness of $[M,N]$ are proved in section 3.

2. Semipotent Rings

Recall that a ring R is a semipotent ring [8] also, called an I_0 −ring by Nicholson [4] and Hakmi [2], if every principal left (resp. right) ideal not contained in $J(R)$ contains a nonzero idempotent, or equivalently, for any $a \in R$, $a \notin J(R)$ there exists $0 \neq x \in R$ such that $x = xax$. Examples of this rings include: (a) Exchange rings (see [5, Proposition 1.9], a ring R is exchange if for each $a \in R$, there exists $e^2 = e \in R$ such that $a - e \in (a^2 - a)R$). (b) Endomorphism rings of injective modules (see [4, Proposition 1.4]). (c) Endomorphism ring of regular modules (in the sense Zelmanowitz [9]), (see [2, Corollary 3.6]).

Proposition 2.1. Let M be a module and $f \in E_M$:

(1) If M is injective and $f \in \nabla(E_M)$ then $f \in \Delta(E_M)$.

(2)If M is projective and E_M is semipotent, then if $f \in \Delta(E_M)$ follows that $f \in \nabla(E_M)$.

Proof. (1). Suppose that M is injective then by [4, Proposition 1.4], $J(E_M) = \Delta(E_M)$. Let $f \in \nabla(E_M)$ and suppose $f \notin J(E_M)$, since E_M is semipotent then $\varphi = \varphi f \varphi$ for some $0 \neq \varphi \in E_M$, thus $\alpha = f\varphi$ is a nonzero idempotent of E_M. Since $Im(\alpha) \subseteq Im(f)$ then $Im(\alpha) << M$, therefore $Im(\alpha) = Im(\alpha) \cap Im(1-\alpha) = 0$ thus $\alpha = 0$ this is a contradiction. Hence $f \in J(E_M) = \Delta E_M$.

(2) Suppose that M is projective and E_M semipotent then by [2, Proposition 3.3], $J(E_M) = \nabla E_M$. Let $f \in \Delta(E_M)$ and suppose $f \notin J(E_M)$, then $\mu = \mu f \mu$ for some $0 \neq \mu \in E_M$, therefore $g = \mu f$ is a nonzero idempotent of E_M and $Ker(f) \subseteq Ker(g)$. Since $f \in \Delta(E_M)$ and $Ker(f) \cap Ker(1-g) = 0$ implies $Ker(1-g) = 0$. Since $1-g$ is an

idempotent in E_M and $M = Im(g) \oplus Im(1-g) = Im(1-g)$ we have $Im(g) = 0$, thus $g = 0$ this is a contradiction. Hence $f \in J(E_M) = \nabla E_M$.

From Proposition 2.1, we derive the following:

Corollary 2.2. Let M be a module:

(1) If M is injective then $\nabla E_M \subseteq \Delta E_M = J(E_M)$.

(2) If M is projective and E_M is semipotent then $\Delta E_M \subseteq \nabla E_M = J(E_M)$.

A ring *R* is called quasi-frobenius [3] if, every projective (injective) module is injective (projective).

Corollary 2.3. For any projective (injective) module M over a quasi-frobenius ring $J(E_M) = \Delta E_M = \nabla E_M$ and E_M is a semipotent ring.

Proof follows immediately from the fact that endomorphism ring of any injective module is semipotent and corollary 2.2.

3. Semipotent [M,N]

Following [8], let M_R, N_R are modules and $[M,N] = hom_R(M,N)$, then $[M,N]$ is an $(E_M, E_N)-$bimodule.

- The Jacobson radical.

$J[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in J(E_M) \text{ for all } \beta \in [N,M]\}$

$J[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in J(E_N) \text{ for all } \beta \in [N,M]\}$

Thus $J[M,M] = J(E_M)$. In particular $J[R,R] = J(R)$.

- The singular ideal.

$\Delta[M,N] = \{\alpha : \alpha \in [M,N];\ Ker(\alpha) \leq_e M\}$

- The co-singular ideal.

$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ Im(\alpha) << N\}$

- The total.

$\text{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ [N,M]\alpha \text{ containsno nonzeroidempotent}\}$

$\text{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha[N,M] \text{ containsno nonzeroidempotent}\}$

Lemma 3.1. [8, Lemma 2.1]. Let M_R, N_R be modules. The following conditions are equivalent:

(1) If $\alpha \in [M,N] \setminus J[M,N]$ there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.

(2) If $\alpha \in [M,N] \setminus J[M,N]$ there exists $\beta \in [N,M]$ such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.

(3) If $\alpha \in [M,N] \setminus J[M,N]$ there exists $\gamma \in [N,M]$ such that $\gamma\alpha\gamma = \gamma \notin J[N,M]$.

Following [8], Recall that $[M,N]$ is semipotent if, the conditions in lemma 3.1 are satisfied. And recalled $\alpha \in [M,N]$ is regular if $\alpha = \alpha\beta\alpha$ for some $\beta \in [N,M]$. $[M,N]$ is called regular if each $\alpha \in [M,N]$ is regular. Thus $[M,M]$ is regular if and only if E_M is regular ring.

Lemma 3.2. $\alpha \in [M,N]$ is regular if and only if $Im(\alpha) \subseteq^{\oplus} N$ and $Ker(\alpha) \subseteq^{\oplus} M$.

Proof. $(\Rightarrow)$. Suppose $\alpha \in [M,N]$ is regular then $\alpha = \alpha\beta\alpha$ for some $\beta \in [N,M]$. Thus $(\beta\alpha)^2 = \beta\alpha \in E_M$ and $Ker(\beta\alpha) = Ker(\alpha)$ therefore $Ker(\alpha) \subseteq^{\oplus} M$. On the other hand, $(\alpha\beta)^2 = \alpha\beta \in E_N$ and $Im(\alpha\beta) = Im(\alpha)$ therefore $Im(\alpha) \subseteq^{\oplus} N$.

$(\Leftarrow)$.Since $Im(\alpha) \subseteq^{\oplus} N$ then there exists $\beta' : Im(\alpha) \rightarrow M$ such that $\alpha(\beta'(x)) = x$ for all $x \in Im(\alpha)$, thus $\alpha(\beta'\alpha(y)) = \alpha(y)$ for all $y \in M$. But $Im(\alpha) \subseteq^{\oplus} N$ so we can extend β' to $\beta \in [N,M]$ by taking $\beta = 0$ on the complementary summand. Then for any $y \in M$, $\alpha\beta\alpha(y) = \alpha(y)$, thus $\alpha = \alpha\beta\alpha$.

Corollary 3.3. $[M,N]$ is regular if and only if for each $\alpha \in [M,N]$, $Im(\alpha) \subseteq^{\oplus} N$ and $Ker(\alpha) \subseteq^{\oplus} M$.

Corollary 3.4. Let P be a projective module. For any module M the following conditions are equivalent:

(1) $[M,P]$ is regular.

(2) For any $\alpha \in [M,P]$, $Im(\alpha) \subseteq^{\oplus} P$.

Proof. $(1) \Rightarrow (2)$. Follows from corollary 3.3. $(2) \Rightarrow (1)$. Let $\alpha \in [M,P]$ then $Im(\alpha) \subseteq^{\oplus} P$, since P is projective then $Ker(\alpha) \subseteq^{\oplus} P$ again by corollary 3.3, $[M,N]$ is regular.

Proposition 3.5. Let M_R, N_R be modules. The following are equivalent:

(1) $[M,N]$ is semipotent.

(2) If $\alpha \in [M,N] \setminus J[M,N]$ there exists $0 \neq \beta \in [N,M]$ such that $Im(\alpha\beta) \subseteq^{\oplus} N$ and $Ker(\alpha\beta) \subseteq^{\oplus} N$.

(3) If $\alpha \in [M,N] \setminus J[M,N]$ there exists $0 \neq \beta \in [N,M]$ such that $Im(\alpha\beta) \subseteq^{\oplus} M$ and $Ker(\alpha\beta) \subseteq^{\oplus} M$.

Proof. Suppose (1) holds. Then $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$ for some $0 \neq \beta \in [N,M]$ thus $Im(\alpha\beta) \subseteq^{\oplus} N$ and $Ker(\alpha\beta) \subseteq^{\oplus} N$. Suppose (2) holds. Then $Im(\alpha\beta) \subseteq^{\oplus} N$ and $Ker(\alpha\beta) \subseteq^{\oplus} N$ for some $0 \neq \beta \in [N,M]$ by [7, Lemma 3.1] there exists $\gamma \in E_N$ such that $(\alpha\beta)\gamma(\alpha\beta) = \alpha\beta$. Thus $\beta\gamma \in [N,M]$ and $0 \neq \alpha(\beta\gamma) = (\alpha(\beta\gamma))^2 \in E_N$ and by lemma 3.1 $[M,N]$ is semipotent. Similarly, the equivalence $(1) \Leftrightarrow (3)$ holds.

Corollary 3.6. Let P be a projective module. For any module M the following condition are equivalent:

(1) $[M,P]$ is semipotent.

(2) For any $\alpha \in [M,P] \setminus J[M,P]$ there exists $0 \neq \beta \in [P,M]$ such that $Im(\alpha\beta) \subseteq^{\oplus} P$.

Proof. $(1) \Rightarrow (2)$ follows immediately from proposition 3.5.

$(2) \Rightarrow (1)$. Let $\alpha \in [M,P] \setminus J[M,P]$ then there exist $0 \neq \beta \in [P,M]$ such that $Im(\alpha\beta) \subseteq^{\oplus} P$ since P is projective then $Ker(\alpha\beta) \subseteq^{\oplus} P$ by proposition 3.5 follows that $[M,P]$ is semipotent.

Proposition 3.7. Let Q be an injective module and P be a projective module. Then:

(1) $[Q,M]$ is semipotent for any $R-$module M.

(2) If E_P is semipotent then $[M,P]$ is semipotent for any $R-$module M.

Proof. (1) Let $\alpha \in [Q,M]$, $\alpha \notin J[Q,M]$ then there exists $\beta \in [M,Q]$ such that $\beta\alpha \notin J(E_Q)$. Since E_Q is a semipotent ring then $\varphi = \varphi\beta\alpha\varphi$ for some $0 \neq \varphi \in E_Q$. Thus $0 \neq (\varphi\beta)\alpha \in E_Q$ is an idempotent and $\varphi\beta \in [M,Q]$ by lemma 3.1, $[M,Q]$ is semipotent.

(2) Suppose that E_P is a semipotent ring. Let $\alpha \in [M,P] \setminus J[M,P]$ then there exists $\beta \in [P,M]$ such that $\alpha\beta \notin J(E_P)$ therefore $\mu = \mu(\alpha\beta)\mu$ for some $0 \neq \mu \in E_P$. Thus $0 \neq \alpha\beta\mu \in E_P$ is an idempotent and $\beta\mu \in [P,M]$ by lemma 3.1, $[M,P]$ is semipotent.

Lemma 3.8. Let M_R, N_R be modules. If $[M,N]$ is semipotent then:

(1) $\Delta[M,N] \subseteq J[M,N]$.

(2) $\nabla[M,N] \subseteq J[M,N]$.

Proof. Suppose that $[M,N]$ is semipotent.

(1) Let $\alpha \in \Delta[M,N]$ then $Ker(\alpha) \leq_e M$. Suppose that $\alpha \notin J[M,N]$ then there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. Since $Ker(\alpha) \subseteq Ker(\beta\alpha)$ then $Ker(\alpha) \cap Im(\beta\alpha) \subseteq Ker(\beta\alpha) \cap Im(\beta\alpha) = 0$. Thus $Im(\beta\alpha) = 0$ and $\beta\alpha = 0$ this is a contradiction. Hence $\alpha \in J[M,N]$.

(2) Let $\alpha \in \nabla[M,N]$ then $Im(\alpha) << N$. Suppose that $\alpha \notin J[M,N]$ then there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. Since

$Im(\alpha) << N$ then $Im(\beta\alpha) << M$ and $Im(\beta\alpha) \subseteq Im(\beta\alpha) \cap Ker(\beta\alpha) = 0$, thus $\beta\alpha = 0$ this is a contradict-ion. Hence $\alpha \in J[M,N]$.

Beidar and Kasch in [1] studied conditions on module P, which imply that Tot$[M,P] = \nabla[M,P] = J[M,P]$ for all $M \in mod-R$ and they showed that these equalities holds if P is semiperfect and projective, the following theorem shows that these equalities hold if P is projective and E_P is semipotent.

Theorem 3.9. Let P be a projective module with E_P is a semipotent ring. Then for any module M_R

$$\Delta[M,P] \subseteq \nabla[M,P] = \mathrm{Tot}[M,P] = J[M,P]$$

In particular, $\Delta E_P \subseteq \nabla E_P = \mathrm{Tot}(E_P) = J(E_P)$.

Proof. Since E_P is a semipotent ring then by proposition 3.7 $[M,P]$ is semipotent and by[8, Theorem 2.2], Tot$[M,P] = J[M,P]$. Let $\alpha \in \nabla[M,P]$ then $Im(\alpha) << P$ therefore $Im(\alpha\beta) << P$ for all $\beta \in [P,M]$, since $J(E_P) = \nabla E_P$ by corollary 2.2, follows $\alpha\beta \in J(E_P)$ thus $\alpha \in J[M,P]$, i.e. $\nabla[M,P] = J[M,P]$.

Let $\alpha \in J[M,P]$ then for any $\beta \in [P,M]$, $\alpha\beta \in J(E_P)$ therefore $Im(\alpha\beta) << P$. Suppose that $Im(\alpha)$ not small in P then by [2, Theorem 3.5] there exists $0 \neq N \subseteq^{\oplus} P$ and $N \subseteq Im(\alpha)$. Let μ be the projection of P onto N then $0 \neq \mu^2 = \mu. \in E_P$. Since $Im(\mu) \subseteq Im(\alpha)$ then $Im(\mu\alpha) = Im(\mu) = N$, no N is projective therefore $Ker(\mu\alpha) \subseteq^{\oplus} P$ and by lemma 3.2, follows that $\mu\alpha \in [M,P]$ is regular, thus there exists $\gamma \in [P,M]$ such that $(\mu\alpha)\gamma(\mu\alpha) = \mu\alpha$ therefore $0 \neq (\mu\alpha\gamma)^2 = \mu\alpha\gamma \in E_P$. Since $\alpha \in J[M,P]$ then $Im(\alpha\gamma) << P$ and $Im(\mu\alpha\gamma) << P$ thus $\mu\alpha\gamma \in J(E_P)$ which contradiction that $J(E_P)$ contains no nonzero idempotent, therefore $Im(\alpha) << P$. This proves $J[M,P] = \nabla[M,P]$. $\Delta[M,P] \subseteq J[M,P]$ follows from lemma 3.8.

Lemma 3.10. Let Q be an injective module. Then for any module M

$$\nabla[Q,M] \subseteq \Delta[Q,M] = \mathrm{Tot}[Q,M] = J[Q,M]$$

Proof. Beidar and Kasch in [1] showed that if Q is injective then $\Delta[Q,M] = \mathrm{Tot}[Q,M] = J[Q,M]$. On the other hand, since Q is injective we have by proposition 3.7, $[Q,M]$ is semipotent and by lemma 3.8, $\nabla[Q,M] \subseteq J[Q,M]$.

Note that, by corollary 2.3; $\mathrm{Tot}(E_M) = J(E_M) = \Delta E_M = \nabla E_M$ for any injective (projective) module over a quasi-frobenius ring. The following Theorem generalize this fact.

Theorem 3.11. Let Q be an injective module and P be a projective module with E_P is a semipotent ring. Then

$$\nabla[Q,P] = \Delta[Q,P] = \mathrm{Tot}[Q,P] = J[Q,P]$$

Proof. By theorem 3.9, we have $\Delta[Q,P] \subseteq \nabla[Q,P] = \mathrm{Tot}[Q,P] = J[Q,P]$ and by lemma 3.10, we have $\nabla[Q,P] \subseteq \Delta[Q,P] = \mathrm{Tot}[Q,P] = J[Q,P]$. Thus $\nabla[Q,P] = \Delta[Q,P] = \mathrm{Tot}[Q,P] = J[Q,P]$.

Recall a projective module P is an I_0-module [2] if, for every submodule K of P, $K \not\subset J(P)$ contains a nonzero direct summand of P.

Theorem 3.12. For any ring R the following conditions are equivalent:

(1) R is a semipotent ring and $J(R)$ is left $T-$nilpotent.

(2) Any projective module $P \in mod-R$ is an I_0- module and $J(P) << P$.

(3) E_P is a semipotent ring for any projective module $P \in mod-R$.

(4) $[M,P]$ is semipotent for any projective module $P \in mod-R$ and any module $M \in mod-R$.

(5) $\mathrm{Tot}[M,P] = J[M,P]$ for any projective module $P \in mod-R$ and any module $M \in mod-R$.

(6) $\text{Tot}[P,N] = J[P,N]$ for any projective module $P \in mod-R$ and any module $N \in mod-R$.

(7) $[P,N]$ is semipotent for any projective module $P \in mod-R$ and any module $N \in mod-R$.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ by [2, Theorem 3.8], (see also [8, Theorem 4.10] for $(1) \Leftrightarrow (3)$). $(3) \Rightarrow (4)$ by proposition 3.7(2). $(4) \Rightarrow (3)$ take $M = P$ for any projective module $P \in mod-R$. $(4) \Leftrightarrow (5)$. By [8, Theorem 2.2]. $(3) \Leftrightarrow (6)$. By [8, Theorem 4.5(2)]. $(6) \Leftrightarrow (7)$. By [8, theorem 2.2].

Remark. In [1] Bediar and Kasch showed that if Q_R is an injective module then $\text{Tot}[Q,N] = J[Q,N]$ for all $N \in mod-R$, so by [8, Theorem 2.2], $[Q,N]$ is semipotent. In particular, if R is self injective then R is a semipotent ring, (see also [6, Theorem 1.3]).

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