قسم الفيزياء - كلية العلوم
جامعة دمشق

السنة الثانية فيزياء
امتحان مادة تحليل متجهي
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سلم تصحيح

المدة: ساعتان
العلامة 100

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$\boxed{10}+\boxed{18}+\boxed{8}=\boxed{36}$ Q I

1- يجب اثبات أن $\vec{\nabla}\times\vec{F}=\vec{0}$ $\boxed{10}$

$$\vec{\nabla}\times\vec{F}=\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}F_x\\ F_y\\ F_z\end{pmatrix}=\begin{pmatrix}\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\\ \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\\ \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\end{pmatrix}$$

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$$=\begin{pmatrix}\frac{x}{y^2z^2}-\frac{x}{y^2z^2}\\ -\frac{1}{yz^2}+\frac{1}{yz^2}\\ -\frac{1}{y^2z}+\frac{1}{y^2z}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\exists\,\phi(x,y,z):\vec{F}=\vec{\nabla}\phi(x,y,z)$$

ايجاد الكمون العددي $\phi(x,y,z)$

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$$\begin{pmatrix}F_x(x,y,z)\\ F_y(x,y,z)\\ F_z(x,y,z)\end{pmatrix}=\begin{pmatrix}\frac{\partial\phi(x,y,z)}{\partial x}\\ \frac{\partial\phi(x,y,z)}{\partial y}\\ \frac{\partial\phi(x,y,z)}{\partial z}\end{pmatrix}$$

$$1+\frac{1}{yz}=\frac{\partial\phi(x,y,z)}{\partial x}\ {}^{1}\Rightarrow\phi(x,y,z)=\int\left(1+\frac{1}{yz}\right)dx=x+\frac{x}{yz}+B(y,z)\ {}^{1}$$

$$-\frac{x}{y^2z}=\frac{\partial\phi(x,y,z)}{\partial y}=-\frac{x}{y^2z}+\frac{\partial B(y,z)}{\partial y}\ {}^{1}\Rightarrow\frac{\partial B(y,z)}{\partial y}=0\Rightarrow B(y,z)=A(z)\ {}^{1}$$

$$\Rightarrow\phi(x,y,z)=x+\frac{x}{yz}+A(z)$$

$$-\frac{x}{yz^2}=\frac{\partial\phi(x,y,z)}{\partial z}=-\frac{x}{yz^2}+\frac{\partial A(z)}{\partial z}\ {}^{1}\Rightarrow\frac{\partial A(z)}{\partial z}=0\Rightarrow A(z)=\ {}^{1}$$

$$\Rightarrow\boxed{\phi(x,y,z)=x+\frac{x}{yz}+C}\ 1$$

$\vec{\nabla}\cdot\vec{\psi}$ $\vec{\nabla}\times\vec{\psi}$
$8 + 10 = 18$

$$\vec{\psi}(r,\theta,\phi) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r\sin\theta\cos\phi \end{pmatrix}$$

$\psi_r = r\cos\theta,\quad \psi_\theta = r\sin\theta,\quad \psi_\phi = r\sin\theta\cos\phi$

$$\vec{\nabla}\cdot\vec{\psi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\psi_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\psi_\theta) + \frac{1}{r\sin\theta}\frac{\partial\psi_\phi}{\partial\phi}$$

$\frac{\partial}{\partial r}(r^3\cos\theta) = 3r^2\cos\theta$ 2 $\qquad \frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) = -r\sin\theta\sin\phi$ 2

$\frac{\partial}{\partial\theta}(r\sin^2\theta) = 2r\sin\theta\cos\theta$ 2

$$\Rightarrow \vec{\nabla}\cdot\vec{\psi} = \frac{1}{r^2}(3r^2\cos\theta) + \frac{1}{r\sin\theta}(2r\sin\theta\cos\theta) + \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi)$$

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$= 3\cos\theta + 2\cos\theta - \sin\theta \Rightarrow \boxed{\vec{\nabla}\cdot\vec{\psi} = 5\cos\theta - \sin\theta}$ [8]

$$\vec{\nabla}\times\vec{\psi} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\psi_\phi\sin\theta) - \frac{\partial\psi_\theta}{\partial\phi}\right]\hat{e}_r + \frac{1}{r\sin\theta}\left[\frac{\partial\psi_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(r\psi_\phi)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\psi_\theta) - \frac{\partial\psi_r}{\partial\theta}\right]\hat{e}_\phi$$

[10]

1 $\qquad$ 1

$\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) = 2r\sin\theta\cos\theta\cos\phi,\quad \frac{\partial}{\partial\phi}(r\sin\theta) = 0$

1 $\qquad$ 1

$\frac{\partial}{\partial\phi}(r\cos\theta) = 0 \qquad \frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) = 2r\sin\theta\cos\phi$

1 $\qquad$ 1

$\frac{\partial}{\partial r}(r^2\sin\theta) = 2r\sin\theta \quad . \quad \frac{\partial}{\partial\theta}(r\cos\theta) = -r\sin\theta$

$$\vec{\nabla}\times\vec{\psi} = \frac{1}{r\sin\theta}\left[2r\sin\theta\cos\phi - 0\right]\hat{e}_r + \frac{1}{r\sin\theta}\left[0 - 2r\sin^2\theta\cos\phi\right]\hat{e}_\theta + \frac{1}{r}\left[2r\sin\theta + r\sin\theta\right]\hat{e}_\phi$$

$$= (2\cos\phi)\hat{e}_r + (-2\sin\theta\cos\phi)\hat{e}_\theta + (3\sin\theta)\hat{e}_\phi$$

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$$\vec{\nabla}\times\vec{\psi} = \begin{pmatrix} 2\cos\phi \\ -2\sin\theta\cos\phi \\ 3\sin\theta \end{pmatrix}$$

$$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$$

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad 2$$

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}}\,\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{x}{x^2+y^2+z^2} \quad 2$$

$$\frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}}\,\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{y}{x^2+y^2+z^2} \quad 2$$

$$\frac{\partial f}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}}\,\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{z}{x^2+y^2+z^2} \quad 2$$

$$\Rightarrow \boxed{df(x,y,z) = \frac{1}{x^2+y^2+z^2}\left(x\,dx + y\,dy + z\,dz\right)}$$

[12] + [18] + [10] + [24] = [64]

QII

[12]

$$f(x,y,z) = x+y, \quad z = F(x,y) = 4 - x^2$$

$$B_2 = [0,2] \times [-5,5] \quad 1$$

$$\frac{\partial F(x,y)}{\partial x} = -2x, \quad 1 \quad \frac{\partial F(x,y)}{\partial y} = 0$$

$$\iint_F f\,d\sigma = \int_0^2\int_0^2 (x+y)\sqrt{4x^2+1}\,dx\,dy \quad 1$$

$$= \int_{-5}^{5} dy \int_0^2 x\sqrt{4x^2+1}\,dx + \int_{-5}^{+5} dy\, y \int_0^2 \sqrt{4x^2+1}\,dx \quad 1$$

$$\int_0^2 x\sqrt{4x^2+1}\,dx = \frac{1}{8}\int \sqrt{t}\,dt = \frac{1}{8}\,\frac{2}{3}\left[t^{3/2}\right]_1^{17} = \frac{1}{12}\left(\sqrt{17^3}-1\right) \quad 3$$

$t(x) = 4x^2+1$

$\Rightarrow dt = 8x\,dx$

$$\int_{-5}^{5} dy = y\Big]_{-5}^{5} = 10, \quad 1 \quad \int_{-5}^{5} y\,dy = \frac{1}{2}y^2\Big]_{-5}^{5} = 0 \quad 1$$

$$\Rightarrow \iint_F f\,d\sigma = (10)\frac{1}{12}\left(\sqrt{17^3}-1\right) + 0 = \frac{5}{6}\left(\sqrt{17^3}-1\right) \quad 2$$

$$\Phi = \iint_{F} \vec{B}\cdot d\vec{a} = \iint_{B_2} \vec{B}\left(\vec{r}(\rho,\phi)\right)\cdot \vec{n}(\rho,\phi)\, d\rho\, d\phi \qquad 1$$

$$\vec{n}(\rho,\phi) := \frac{\partial \vec{r}(\rho,\phi)}{\partial \rho} \times \frac{\partial \vec{r}(\rho,\phi)}{\partial \phi} \qquad 1$$

المعادلات البارامترية للسطح

$$\vec{r}(\rho,\phi) = \begin{pmatrix} \rho\cos(\phi) \\ \rho\sin(\phi) \\ 0 \end{pmatrix}^{2}, \quad \rho \in [0,R]^{1}, \quad \phi \in [0,2\pi]^{1}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}^{1}, \quad \frac{\partial \vec{r}}{\partial \phi} = \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix}^{1}$$

$$\Rightarrow \vec{n} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix} = {}^{2}\begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}, \quad \vec{B}\left(\vec{r}(\rho,\phi)\right) = {}^{1}\begin{pmatrix} -1 \\ 1 \\ \rho^2\cos^2\phi \end{pmatrix}$$

$$\Rightarrow \Phi = \int_0^{2\pi}\int_0^{R} \begin{pmatrix} -1 \\ 1 \\ \rho^2\cos^2\phi \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} d\rho\, d\phi = \int_0^{2\pi}\int_0^{R} \rho^3\cos^2\phi\, d\rho\, d\phi \qquad 3$$

$$= \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^{R} \rho^3\, d\rho$$

$$\int_0^{R} \rho^3\, d\rho = \frac{1}{4}\rho^4 \Big]_0^{R} = \frac{R^4}{4} \qquad 1$$

$$\int_0^{2\pi} \cos^2\phi\, d\phi = \int_0^{2\pi}\left[\frac{1}{2} + \frac{1}{2}\cos 2\phi\right] d\phi = \pi + 0 = \pi \qquad 2$$

$$\Rightarrow \boxed{\Phi = \frac{1}{4}\pi R^4} \qquad 1$$

نص مبرهنة Gauß $[5]+[5]=[10]$

إن التكامل السطحي لحقل شعاعي مستمر وقابل للمفاضلة في الفضاء على سطح مغلق يساوي التكامل الحجمي لتفرق الحقل الشعاعي على الحجم الفراغي المحدود بذلك السطح المغلق 2

$$\oiint_{F} \vec{F}\cdot d\vec{\sigma} = \iiint_{B_3} (\vec{\nabla}\cdot\vec{F})\, d\tau \qquad 1$$

$F$ السطح المغلق

$B_3$ الحجم الفراغي المحدود ب $F$ 2

$d\vec{\sigma}$ عنصر السطح ، $d\tau$ عنصر الحجم .

$\vec{F}\cdot d\vec{\sigma}$ تدفق الحقل الشعاعي $\vec{F}$ . $\vec{\nabla}\cdot\vec{F}$ تفرق الحقل الشعاعي $\vec{F}$

ii- $[5]$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \iiint_{B_3} \frac{\partial \rho}{\partial t}\, d\tau + \iiint_{B_3} (\vec{\nabla}\cdot\vec{J})\, d\tau = 0 \Rightarrow \qquad 1$$

$$\frac{\partial}{\partial t}\iiint_{B_3} \rho\, d\tau + \oiint_{F} \vec{J}\cdot d\vec{\sigma} = 0 \Leftrightarrow \frac{\partial q(t)}{\partial t} + I(t) = 0 \qquad 1$$

2

إن تغير كمية الشحنة الكهربائية $q(t)$ في منطقة $B_3$ تساوي كمية الشحنة الكهربائية التي تمر عبر السطح $F$ للمنطقة $B_3$ .

$$\vec{F}(x,y,z) = \begin{pmatrix} x^3 \\ y \\ 1+z^2 \end{pmatrix} , \quad F \in R^3$$

$$\Phi := \oiint_{S} \vec{F}\cdot d\vec{\sigma} = \iiint_{B_3} (\vec{\nabla}\cdot\vec{F})\, d\tau \qquad 1$$

$$\vec{\nabla}\cdot\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x^3 \\ y \\ 1+z^2 \end{pmatrix} = 3x^2 + 1 + 2z \qquad 2$$

$$B_3 = \{(\rho,\phi,z)\in \mathbb{R}^3 : 0\le\rho\le 2,\ 0\le\phi\le 2\pi,\ 0\le z\le 3\} \quad 1$$

$$d\tau = dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz\ ,\quad \rho=\sqrt{x^2+y^2}\ ,\quad x=\rho\cos\phi$$

1 1 1

$$\Rightarrow \vec{\nabla}\cdot\vec{F} = 3\rho^2\cos^2\phi + 1 + 2z \quad 2$$

$$\Rightarrow \Phi = \int_0^3 dz\int_0^2 \rho\,d\rho\int_0^{2\pi} d\phi\,(3\rho^2\cos^2\phi+1+2z) \quad 2$$

$$= \underbrace{3\int_0^3 dz\int_0^2 \rho^3 d\rho\int_0^{2\pi} d\phi\cos^2\phi}_{I_1} + \underbrace{\int_0^3 dz\int_0^2 \rho\,d\rho\int_0^{2\pi} d\phi}_{I_2} + \underbrace{2\int_0^3 dz\,z\int_0^2 \rho\,d\rho\int_0^{2\pi} d\phi}_{I_3} \quad 1\ 1\ 1$$

$$I_1 = \int_0^3 dz\int_0^2 \rho^3 d\rho\int_0^{2\pi} d\phi\cos^2\phi = \int_0^3 dz\int_0^2 \rho^3 d\rho\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2\phi\right)d\phi$$

$$= \int_0^3 dz\int_0^2 \rho^3 d\rho\left(\underbrace{\frac{\phi}{2}\Big|_0^{2\pi}}_{=\pi} + \underbrace{\frac{1}{4}\sin 2\phi\Big|_0^{2\pi}}_{=0}\right)$$

4

$$= \pi\int_0^3 dz\left(\frac{\rho^4}{4}\Big|_0^2\right) = \pi\,\frac{16}{4}\int_0^3 dz = \pi\,\frac{16}{4}\left(z\Big|_0^3\right) = \pi$$

$$\Rightarrow \boxed{I_1 = 12\pi}$$

$$2\quad I_2 = \int_0^3 dz\int_0^2 \rho\,d\rho\int_0^{2\pi} d\phi = \left(z\Big]_0^3\right)\left(\frac{\rho^2}{2}\Big]_0^2\right)\left(\phi\Big]_0^{2\pi}\right) = (3)\left(\frac{4}{2}\right)(2\pi) = 1$$

$$2\quad I_3 = \int_0^3 dz\,z\int_0^2 \rho\,d\rho\int_0^{2\pi} d\phi = \left(\frac{1}{2}z^2\Big]_0^3\right)\left(\frac{\rho^2}{2}\Big]_0^2\right)\left(\phi\Big]_0^{2\pi}\right) = \frac{9}{2}\,\frac{4}{2}\,2\pi = 1$$

$$\Rightarrow \Phi = 3I_1 + I_2 + 2I_3$$

$$= 3(12\pi) + (12\pi) + 2(18\pi) = 36\pi + 12\pi + 36\pi \Rightarrow$$

$$2\quad \boxed{\Phi = 84\pi}$$